BOUNDS ON THE TCT TYPE SET OF $\mathrm{V}(\mathbf{A})$, A IDEMPOTENT

RALPH FREESE

Let $\mathbf{A}$ be a finite idempotent algebra. We present here the program used by UACalc to calculate upper and lower bounds on $\operatorname{typ}(\boldsymbol{V}(\mathbf{A}))$, the TCT type set of $\boldsymbol{V}(\mathbf{A})$. Let

$$\begin{aligned} S^* &= \operatorname{typ}(\boldsymbol{S}(\mathbf{A})) \\ S &= \operatorname{typ}\{\mathbf{B} \in \boldsymbol{HS}(\mathbf{A}) : \mathbf{B} \text{ is strictly simple}\} \\ T &= \operatorname{typ}(\mathbf{A}) \\ U &= S \cup T \end{aligned}$$

By basic Tame Congruence Theory we have $S \subseteq S^*$. Lemma 2.3 and Theorem 6.3 of [2] show how to compute S very quickly. On the other hand computing S^* might be exponential in time.

The Prorgram:

(1) If $T = \{\mathbf{1}\}$ then $\operatorname{typ}(\boldsymbol{V}(\mathbf{A})) = \{\mathbf{1}\}$.
(2) If $T = \{\mathbf{1}, \mathbf{2}\}$ then $\operatorname{typ}(\boldsymbol{V}(\mathbf{A})) = \{\mathbf{1}, \mathbf{2}\}$.
(3) If $T = \{\mathbf{2}\}$,
  (a) if $\mathbf{1} \in S$ then $\operatorname{typ}(\boldsymbol{V}(\mathbf{A})) = \{\mathbf{1}, \mathbf{2}\}$.
  (b) If not, then $\operatorname{typ}(\boldsymbol{V}(\mathbf{A})) = \{\mathbf{2}\}$.
(4) If $\mathbf{1} \in U$ then $U \subseteq \operatorname{typ}(\boldsymbol{V}(\mathbf{A})) \subseteq \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$.
(5) If $\mathbf{5} \in U$,
  (a) if $\mathbf{2} \in U$, then $U \subseteq \operatorname{typ}(\boldsymbol{V}(\mathbf{A})) \subseteq \{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$.
  (b) If not, $U \subseteq \operatorname{typ}(\boldsymbol{V}(\mathbf{A})) \subseteq \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$.
(6) If $\mathbf{3} \in U$ then $\operatorname{typ}(\boldsymbol{V}(\mathbf{A})) = U$.
(7) If $\boldsymbol{V}(\mathbf{A})$ is congruence modular, then $\operatorname{typ}(\boldsymbol{V}(\mathbf{A})) = S^*$.
(8) Otherwise, $U \subseteq \operatorname{typ}(\boldsymbol{V}(\mathbf{A})) \subseteq U \cup \{\mathbf{3}\}$.

Justification. The main tool is Valeriote's less-equal theorem [5]; see Proposition 2.1 of [2]:

Theorem 1. *If* $\mathbf{i} \in \operatorname{typ}(\boldsymbol{V}(\mathbf{A}))$, $\mathbf{A}$ *idempotent, then there is a* $\mathbf{j} \in S$ *with* $\mathbf{j} \leq \mathbf{i}$ *under the usual ordering of types:* $\mathbf{1} < \mathbf{2} < \mathbf{3}$ *and* $\mathbf{1} < \mathbf{5} < \mathbf{4} < \mathbf{3}$.

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Items (1), (2) and (3) use the following result of Hobby and McKenzie; see Proposition 3.1 of [1].

Theorem 2. *Let* $\mathbf{A}$ *be finite. Then*

(1) *If* $T = \{\mathbf{1}\}$ *then* $\mathrm{typ}(\boldsymbol{V}(\mathbf{A})) = \{\mathbf{1}\}$.
(2) *If* $T \subseteq \{\mathbf{1}, \mathbf{2}\}$ *then* $\mathrm{typ}(\boldsymbol{V}(\mathbf{A})) \subseteq \{\mathbf{1}, \mathbf{2}\}$

Item (7) uses the following which follows from the results of [3]; see also [4].

Theorem 3. *If* $\mathbf{A}$ *is a finite algebra in a congruence modular variety, then* $\mathrm{typ}(\boldsymbol{V}(\mathbf{A})) = S^*$.

The algorithm pushes finding S^* as far down as possible since it can be exponential time. Of course we could skip that step and the result would be slightly weaker but still correct. Notice that if we get to (8), $U = \{\mathbf{4}\}$ or $\{\mathbf{2}, \mathbf{4}\}$ and the only question remaining is "is $\mathbf{3}$ also in $\mathrm{typ}(\boldsymbol{V}(\mathbf{A}))$."

References

[1] J. Berman, E. Kiss, P. Prőhle, and Á. Szendrei, *The type set of a finitely generated variety*, Discrete Math. **112** (1993), 1–20.

[2] R. Freese and M. Valeriote, *On the complexity of some Maltsev conditions*, Intern. J. Algebra & Computation **19** (2009), no. 1, 41–77.

[3] Ralph Freese, *Subdirectly irreducible algebras in modular varieties*, Universal Algebra and Lattice Theory (R. Freese and O. Garcia, eds.), Springer-Verlag, New York, 1983, Lecture notes in Mathematics, vol. **1004**, pp. 142–152.

[4] Emil W. Kiss and Péter Pröhle, *Problems and results in tame congruence theory. A survey of the '88 Budapest Workshop*, Algebra Universalis **29** (1992), no. 2, 151–171.

[5] Matthew A. Valeriote, *A subalgebra intersection property for congruence distributive varieties*, Canad. J. Math. **61** (2009), no. 2, 451–464.