A SIMPLER MALTSEV CONDITION FOR LOCALLY FINITE DIFFERENCE TERM VARIETIES

Following [3, Def. 5.2], given congruences α, β, γ of an algebra, define $\tau(\alpha, \beta, \gamma)$ to be the transitive closure of $\beta \cup (\alpha \cap (\gamma \circ (\alpha \cap \beta) \circ \gamma))$.

The next theorem is shown by combining results of Kearnes and Szendrei [2, 3]. It slightly improves an observation made in passing at the end of section 1 in [4].

Theorem 0.1. *Let $\mathcal{V}$ be a variety for which $\mathbf{F}_{\mathcal{V}}(2)$ is finite. The following are equivalent:*

(1) *$\mathcal{V}$ has a difference term.*
(2) *For all $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$,*

$$\alpha \cap (\beta \circ \gamma) \subseteq (\alpha \cap \tau) \circ \gamma \circ \beta$$

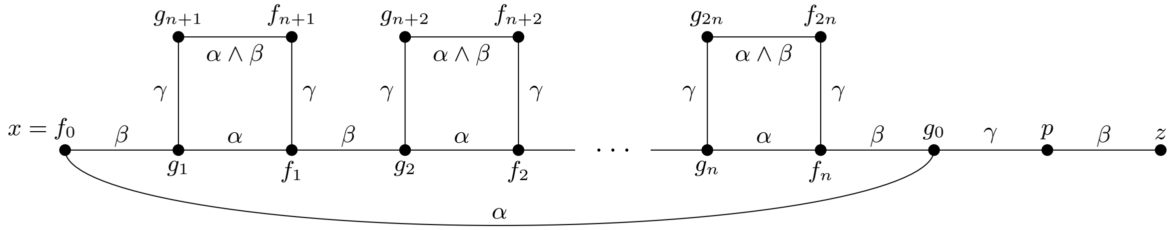
where $\tau = \tau(\alpha, \beta, \gamma)$.
(3) *For some n there exist idempotent terms $f_i(x, y, z), g_i(x, y, z)$, $0 \leq i \leq 2n$, and $p(x, y, z)$ such that the following are identities of $\mathcal{V}$:*
   (a) $f_0(x, y, z) \approx x$
   (b) $f_i(x, y, x) \approx g_i(x, y, x)$ *for all i.*
   (c) $f_i(x, x, y) \approx g_{i+1}(x, x, y)$ *for $i < n$ and* $f_n(x, x, y) \approx g_0(x, x, y)$.
   (d) $f_{n+i}(x, x, y) \approx g_{n+i}(x, x, y)$ *for $1 \leq i \leq n$.*
   (e) $f_i(x, y, y) \approx f_{n+i}(x, y, y)$ *and* $g_i(x, y, y) \approx g_{n+i}(x, y, y)$ *for $1 \leq i \leq n$.*
   (f) $g_0(x, y, y) = p(x, y, y)$ *and* $p(x, x, y) \approx y$.

Moreover, if f_i, g_i, p are terms satisfying the identities in (3), then p is a difference term for $\mathcal{V}$ and the pairs (f_i, g_i) witness [4, Theorem 1.2(3)].

Proof. (2) $\Leftrightarrow$ (3) is established in the standard way, and (2) $\Rightarrow$ (1) follows from [4, Theorem 1.2 (2) $\Rightarrow$ (1)] because $\tau \leq \beta_2 := \beta \vee (\alpha \wedge (\gamma \vee (\alpha \wedge \beta)))$.

To prove (1) $\Rightarrow$ (3), we can replace $\mathcal{V}$ by the subvariety of $\mathcal{V}$ generated by $\mathbf{F}_{\mathcal{V}}(2)$, since the identities in (3) are two-variable identities. Thus we can assume that $\mathcal{V}$ is locally finite. Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y, z)$ and as usual let $\alpha = \mathrm{Cg}^{\mathbf{F}}(x, z)$, $\beta = \mathrm{Cg}^{\mathbf{F}}(x, y)$ and $\gamma = \mathrm{Cg}^{\mathbf{F}}(y, z)$. Also let $\tau = \tau(\alpha, \beta, \gamma)$. By finiteness of $\mathbf{F}$ and [3, Lemma 5.3], $\alpha - \tau$ contains no 2-snags, so α is solvable over $\alpha \wedge \tau$ by [1, Theorem 7.2]. Hence there exists $m \geq 0$ such that so that $[\alpha]^m \leq \tau$.

By [2, Lemma 2.7], there exists a term $p(x, y, z)$ such that $\mathcal{V} \models p(x, x, y) \approx y$ and $(x, p^{\mathbf{F}}(x, z, z)) \in [\alpha]^m$. Hence $(x, p^{\mathbf{F}}(x, z, z)) \in \alpha \cap \tau$, so $(x, z) \in (\alpha \cap \tau) \circ \gamma \circ \beta$ witnessed by $p^{\mathbf{F}}(x, z, z)$ and $p^{\mathbf{F}}(x, y, z)$. The identities then follow in the standard way, and the "Moreover" claim follows from arguments in [4, Theorem 1.2]. $\square$



References

[1] David Hobby and Ralph McKenzie, *The Structure of Finite Algebras*. Contemp. Math. **76**, Amer. Math. Soc., 1988.

[2] Keith A. Kearnes, Varieties with a difference term, *J. Algebra* **177** (1995), 926–960.

[3] Keith A. Kearnes and Ágnes Szendrei, The relationship between the two commutators, *Internat. J. Algebra Comput.* **8** (1998), 492–531.

[4] Keith Kearnes, Ágnes Szendrei and Ross Willard, A finite basis theorem for difference-term varieties with a finite residual bound, *Trans. Amer. Math. Soc.* **368** (2016), 2115–2143.