NOTES ON CENTRALITY RELATIONS, TERM CONDITIONS, AND COMMUTATORS

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Let $\mathbf{A}$ be an algebra and let S and T be tolerances on $\mathbf{A}$. Let $M(S,T)$, or $M^{\mathbf{A}}(S,T)$ to emphasize $\mathbf{A}$, be the set of all 2×2 matrices of the form

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} f(\mathbf{a},\mathbf{u}) & f(\mathbf{a},\mathbf{v}) \\ f(\mathbf{b},\mathbf{u}) & f(\mathbf{b},\mathbf{v}) \end{bmatrix} \tag{1}$$

where $f(\mathbf{x},\mathbf{y})$ is an $(m+n)$-ary polynomial of $\mathbf{A}$, $\mathbf{a}\ S\ \mathbf{b}$, and $\mathbf{u}\ T\ \mathbf{v}$ (componentwise, of course). The members of $M(S,T)$ are called S,T-*matrices.*

The first exercise gives an efficient way to find $M(S,T)$.

Exercises

1. Show that $M(S,T)$ is the subalgebra of $\mathbf{A}^4$ generated by

$$\left\{ \begin{bmatrix} a & a \\ b & b \end{bmatrix} : a\ S\ b \right\} \cup \left\{ \begin{bmatrix} c & d \\ c & d \end{bmatrix} : c\ T\ d \right\}$$

2. Use the symmetry of S and T to show the matrix obtained from an S,T-matrix by interchanging the rows or columns (or both) is also in $M(S,T)$.

3. $M(T,T)$ is closed under taking transposes.

Centrality Relations

We define four kinds of centrality, called centrality, strong centrality, weak centrality, and strong rectularity. The is a fifth centrality condition known as rectangularity which we will save for later.

Let δ be a congruence and S and T be tolerance relations on $\mathbf{A}$. The above centrality relations are denoted $\mathbf{C}(S,T;\delta)$ (centrality), $\mathbf{S}(S,T;\delta)$ (strong centrality), $\mathbf{W}(S,T;\delta)$ (weak centrality), and $\mathbf{S}R(S,T;\delta)$ (strong rectangularity). They hold if the appropriate implication below holds for all

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(S,T)$$

- centrality: $p\ \delta\ q \implies r\ \delta\ s$.
- strong rectangularity: $p\ \delta\ s \implies r\ \delta\ s$.
- weak centrality: $p\ \delta\ q\ \delta\ s \implies r\ \delta\ s$.
- strong centrality holds if both centrality and strong rectangularity hold.

Using the exercises it is easy to see that the implication defining $\mathbf{C}(S, T; \delta)$ can be replaced by $r\ \delta\ s \implies p\ \delta\ q$ and this is equivalent to

$$p\ \delta\ q \Longleftrightarrow r\ \delta\ s.$$

Similar statements hold for the other conditions: weak centrality is equivalent to saying that if any three of p, q, r and s are δ related, then they all are. And strong rectangularity says that if the elements of the main diagonal, or of the sinister diagonal, are δ related, then all four are.

The *S, T-term condition* is the condition $\mathbf{C}(S, T, 0)$, usually expressed using the right-hand matrix in (1). Other kinds of term conditions are defined similarly.

If $\mathbf{C}(S, T; \delta_i)$ holds for all $i \in I$, then $\mathbf{C}(S, T; \bigwedge_{i\in I} \delta_i)$ holds. Similar statements hold for the other centrality conditions. So there is a least δ such that $\mathbf{C}(S, T; \delta)$ holds. This δ is the *commutator* of S and T, and is denoted $[S, T]$. The commutators for the other centrality relations are denoted $[S, T]_{\mathbf{S}}$, $[S, T]_{\mathbf{SR}}$, and $[S, T]_{\mathbf{W}}$.

The properties of these centrality relations are coverered in Theorem 2.19 and Theorem 3.4 of [3]. Much stronger properties hold in congruence modular varieties; see [1].

Exercises

1. As defined in [2], β is *strongly Abelian* over δ ($\delta \le \beta$, both congruences on $\mathbf{A}$) if the following implication holds for all polynomials f and all elements $x_0, \ldots, x_{n-1}$, $y_0, \ldots, y_{n-1}$, and $z_1, \ldots, z_{n-1}$ with $x_0\ \beta\ y_0$ and $x_i\ \beta\ y_i\ \beta\ z_i$, $i = 1, \ldots n-1$.

$$\begin{aligned} &f(x_0, \ldots, x_{n-1})\ \delta\ f(y_0, \ldots, y_{n-1}) \\ &\qquad \implies f(x_0, z_1, \ldots, z_{n-1})\ \delta\ f(y_0, z_1, \ldots, z_{n-1}) \end{aligned}$$

Show that β is strongly Abelian over δ if and only if $\mathbf{S}(\beta, \beta; \delta)$ holds, and also show this is in turn equivalent to $\mathbf{SR}(\beta, \beta; \delta)$.

References

[1] Ralph Freese and Ralph McKenzie, *Commutator theory for congruence modular varieties*, London Mathematical Society Lecture Note Series, vol. 125, Cambridge University Press, Cambridge, 1987, Online version available at: http://www.math.hawaii.edu/~ralph/papers.html.

[2] D. Hobby and R. McKenzie, *The structure of finite algebras (tame congruence theory)*, Contemporary Mathematics, American Mathematical Society, Providence, RI, 1988.

[3] Keith A. Kearnes and Emil W. Kiss, *The shape of congruence lattices*, Mem. Amer. Math. Soc. **222** (2013), no. 1046, viii+169.

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